COBEM 99
XV Congresso Brasileiro de Engenharia Mecânica
15th Brazilian Congress of Mechanical Engineering
22 - 26 de Novembro de 1999 / November 22 - 26, 1999 Águas de Lindóia, São Paulo,

NONLINEAR DYNAMICS OF BEAMS: REDUCED MATHEMATICAL MODELS

João C. André
Escola Politécnica da Universidade de São Paulo
São Paulo, SP, Brasil
andrejc@usp.br

Daniel S. Pereira
Escola Politécnica da Universidade de São Paulo

Abstract. *This work addresses various ideas on the formulation of a reduced mathematical model to the vibration of Bernoulli-Euler beams. The hypothesis of constant axial deformation* $\bar{\varepsilon}$ *within each element allows the use of a single independent function (in this paper, the axis rotation* θ *) to generate the displacement field. This technique is used on simply supported beams with pinned and roller end, applying nonconventional superposition procedures to obtain the nonlinear equations of motion associated to the nonlinear vibration of the first mode, both about the deformed and undeformed configuration. Results with various numbers of elements are compared to those in the literature.*

Keywords: *Non-linear dynamics, Reduction, Beam, Free vibration, Superposition*

1. INTRODUCTION

The establishment of nonlinear equations of motion referred to a few number of generalized coordinates which still represent the physical behavior of the structure constitutes a challenging research field. Its importance is related to the possibility of performing parametric and instability analytical studies, which are practically impossible with large degrees of freedom. Therefore, it is not enough to discretise the structure. One has to drastically reduce the number of degrees of freedom. This procedure is called reduction technique.

The authors have applied reduction techniques (André, Mazzilli & Pereira, 1999; André & Mazzilli, 1996; and André, 1996) based on the definition of a nonlinear displacement field about the deformed and undeformed configuration of equilibrium. The nonlinear displacement field is obtained by superposing the equilibrium displacement field of a nonlinear static analysis and the nonlinear displacement field of a combination of selected natural modes and nonlinear terms derived from subsidiary conditions. Usually, one obtains the equilibrium displacement field and the natural modes and frequencies by the finite element method. This non-conventional superposition procedure has been successfully applied by the authors to the nonlinear dynamic analysis of linear elastic beams and planar frames under holonomic constraints and under the action of static and dynamic forces and support motions.

Several mathematical models may be derived from the Bernoulli-Euler beam theory. This work presents one of these models, and applies it to obtain explicit nonlinear equations of motion of the first natural mode both about the reference and deformed configuration of a simply supported beam.

2. SOME RELEVANT EXPRESSIONS OF THE BERNOULLI-EULER BEAM THEORY

Figure 1 presents geometrical parameters which define the transformation from the reference configuration to the deformed configuration.

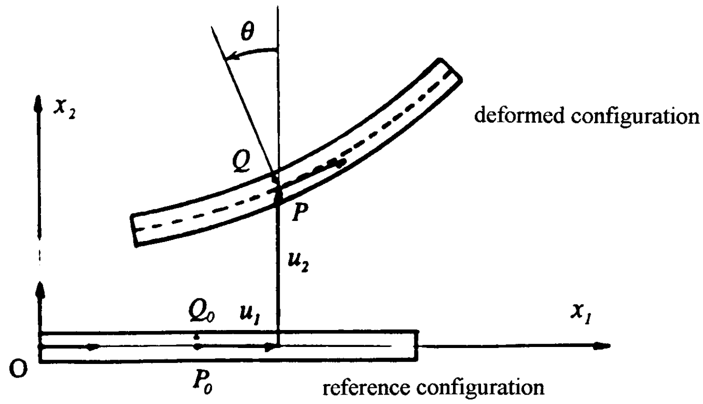

Figure 1 – Reference and deformed configuration of a beam

After some transformations, one may obtain the following expressions:

$$sin\theta = \frac{u_2'}{\lambda} \quad , \quad \cos\theta = \frac{1+u_1'}{\lambda} \quad , \quad \tan\theta = \frac{u_2'}{1+u_1'} \tag{1}$$

where λ is the stretching and is given by:

$$\lambda^2 = (1+u_1')^2 + (u_2')^2 \tag{2}$$

The exact expression of the strain energy for engineering stress (proportional to engineering strain) and for a displacement formulation is given by:

$$U = \frac{1}{2}\int EI(\theta')^2\, dx + \frac{1}{2}\int ES(\lambda - 1)^2\, dx \tag{3}$$

where θ' is directly obtained from θ and x is the coordinate along the reference configuration of the beam. The gravitational potential energy is

$$P = -g\int m u_2 dx \tag{4}$$

where m is the linear density (mass per length). The kinetic energy is

$$K = \frac{1}{2}\int m\left(\dot{u}^2 + \dot{v}^2\right)dx \tag{5}$$

One may see from (1) and (2) that only two of the related functions $u_1'(x)$, $u_2'(x)$ and $\theta(x)$ define the displacement field in a Bernoulli-Euler beam. The introduction of a subsidiary condition allows the displacement field to be defined by a single function. We notice that $u_1(x)$ is not an option for an independent function, because it is of higher order than $u_2(x)$ or $\theta(x)$ in any expansion of (1) or (2).

3. A DERIVED NONLINEAR MODEL FROM BERNOULLI-EULER BEAM THEORY

We assume that the longitudinal deformation ε is constant in each element. Therefore, the displacement field is defined by either $\theta(x)$ or $u_2(x)$. We consider that $\theta(x)$ is the best option for the independent function.

$$\varepsilon = \lambda - 1 \approx u_1' + \frac{1}{2}\left(u_2'\right)^2 \cong u_1' + \frac{1}{2}\theta^2 \tag{6}$$

The medium value of ε for an element of length ℓ starting at x_0 is

$$\bar{\varepsilon} = \frac{1}{h}\left[u_1\left(x_0 + \ell\right) - u_1\left(x_0\right) + \frac{1}{2}\int_{x_0}^{x_0+\ell}\theta^2 dx\right] \tag{7}$$

Expanding the first two expressions of Eq. (1) on θ, we establish the expressions for longitudinal and transverse displacements on the bar axis

$$\begin{aligned} u_1' &= \left(\bar{\varepsilon} + 1\right)\left(1 - \frac{\theta^2}{2} + \frac{\theta^4}{24} + \mathbf{O}\left(\theta^6\right)\right) - 1 \\ u_2' &= \left(\bar{\varepsilon} + 1\right)\left(\theta - \frac{\theta^3}{6} + \mathbf{O}\left(\theta^5\right)\right) \end{aligned} \tag{8}$$

Considering Eq. (8), we notice that the definition of θ allows us to determine the nonlinear displacement field.

Once we intend to analyze the effects of geometrical nonlinearities on the first vibration mode, the nonlinear displacement field is defined by one single generalized displacement. The dynamic rotation at the left end of the beam, q, was chosen as modal coordinate, and the total displacement is the superposition of equilibrium dynamic displacements, meaning

$$\begin{aligned} u_1(x,t) &= u_1^{\text{eq}}(x) + q(t)u_1^{\text{mode}}(x) \\ u_2(x,t) &= u_2^{\text{eq}}(x) + q(t)u_2^{\text{mode}}(x) \\ \theta(x,t) &= \theta^{\text{eq}}(x) + q(t)\theta^{\text{mode}}(x) \end{aligned} \tag{9}$$

Once the displacement field is determined, the process to obtain the nonlinear equation of motion is quite straightforward and can be summarized as follows.

First, the deformed configuration and the first linear mode about both reference and deformed configuration are obtained; the authors used Adina finite-element system and 2-node elements. Those results were used to generate the a rotation function θ for each element,

$$\theta(x) = p_2 N_2(x) + p_3 N_3(x) + p_5 N_5(x) + p_6 N_6(x) \tag{10}$$

where $N_i(x)$ are quadratic interpolation polynomials generated by first order static analysis procedures and $p_i = p_i^{\mathrm{eq}} + q(t) \cdot p_i^{\mathrm{mode}}$ are the displacements at the ends of the element.

Once θ and then the resulting $\bar{\varepsilon}(t)$ according to Eq. (7) are determined for each element, $u_1'(x)$ and $u_2'(x)$ according to Eq. (8) can be found and u_1 and u_2 are determined by algebraic integration. Finally, the expressions for strain, potential and kinetic energy can be written and the total energy of the bar will be the sum of the energy of each element. The Lagrangean is then determined, and the nonlinear equation of motion is defined by the Euler-Lagrange equation, and is given by

$$\ddot{q} + \alpha\left(q\dot{q}^2 + q^2\ddot{q}\right) + \beta\dot{q}^2 + \xi q^2 + \gamma q^3 + \omega^2 q = 0 \tag{11}$$

4. CASE STUDY

The process was applied for the simply-supported beams shown on Figure 2 and Figure 3. For each one of those beams, the equations were obtained under two sets of conditions: vibrations about the reference configuration disregarding gravity influences and vibrations about the deformed configuration under self weight load.

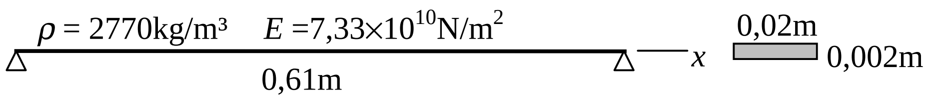

Figure 2 – pinned-pinned beam: physical and geometrical characteristics

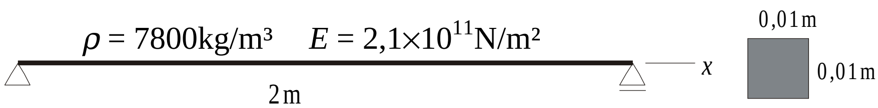

Figure 3 – pinned-roller beam: physical and geometrical characteristics

A thorough evaluation of the results would only be possible after the integration of the equations herein obtained, which is out of the scope of this work, but some of the coefficients of the main terms of those equations are presented in tables 1 through 4.

Null coefficients are omitted and vibrations about the reference configuration are compared to the results of Soares (1998). As a measure of the global convergence of the model, the ratio between the equilibrium modal coordinate obtained (theoretically null) and the equilibrium rotation at the beam end is also presented for the vibrations about the deformed configuration.

Table 1 – Pinned-end beam equation coefficients: first mode about reference configuration

		ω^2	γ	α	ω *(rad/s)*
Soares		6,2055E3	4,654E9	2,4847E0	78,77
André & Pereira	**2 el**	6,254E3	1,729E8	2,18E-1	79,09
	4 el	6,209E3	2,467E8	7,80E-2	78,80
	6 el	6,207E3	2,555E8	3,82E-2	78,78
	8 el	6,206E3	2,588E8	2,31E-2	78,78
	16 el	6,232E3	2,621E8	9,42E-3	78,94

Table 2 – Pinned-end beam equation coefficients: first mode about deformed configuration

		ω^2	ξ	β	γ	α	ω *(rad/s)*	$q_{eq}/\ \theta_{max}$
André & Pereira	**2 el**	2,598E4	-2,949E6	-1,545E-3	1,484E8	2,326E-1	161,2	10%
	4 el	2,309E4	-2,915E6	-5,039E-4	2,440E8	7,398E-2	151,9	2,5%
	6 el	2,268E4	-2,925E6	-2,747E-4	2,555E8	3,253E-2	150,6	1,1%
	8 el	2,255E4	-2,928E6	-1,204E-4	2,593E8	2,134E-2	150,2	0,61%
	16 el	2,244E4	-2,930E6	7,147E-7	2,629E8	6,992E-3	149,8	0,32%

Table 3 – Roller-end beam equation coefficients: first mode about reference configuration

		ω^2	γ	α	ω *(rad/s)*
Soares		1,3659E3	-3,0057E2	-2,2541E0	36,95
André & Pereira	**2 el**	1,3767E3	1,64E7	2,16E-1	37,10
	4 el	1,3667E3	2,33E7	7,34E-2	36,97
	6 el	1,3661E3	2,42E7	3,18E-2	36,96
	8 el	1,3661E3	2,45E7	1,80E-2	36,96
	16 el	1,3667E3	2,48E7	3,42E-3	36,97

Table 4 – Roller-end beam equation coefficients: first mode about deformed configuration

		ω^2	ξ	β	γ	α	ω *(rad/s)*	q_{eq}/θ_{max}
André & Pereira	**2 el**	2,20E3	-1,3E5	1,6E-3	1,636E7	2,7E-1	46,9	4,9%
	4 el	1,49E3	-2,0E4	4,5E-4	2,333E7	7,9E-2	38,6	0,79%
	6 el	1,41E3	-8,6E3	1,3E-4	2,417E7	3,2E-2	37,6	0,17%
	8 el	1,39E3	-4,7E3	2,5E-4	2,448E7	2,1E-2	37,3	0,081%
	16 el	1,37E3	-1,1E3	3,3E-2	2,479E7	5,0E-3	37,1	0,39%

As expected, the geometric non-linearities sensibly increased the rigidity of the pinned-pinned beam in the vibrations about the deformed configuration, which did not happen at all in the pinned-roller beam.

5. CONCLUDING REMARKS

The main goal in this procedure is to add non-linear behavior to the equation of free vibrations of a simply supported beam through the influence of flexural deformations on longitudinal displacements and longitudinal deformation. This is accomplished by a twofold approach: first and foremost, by not neglecting the stretching in both longitudinal and transverse displacements, and second, by not neglecting the longitudinal displacements in the kinetic energy. However, it should be stated that such an idea might not be very well suited to assess vibrations about the reference configuration, once there is no longitudinal displacement in those modes, and therefore the resulting function u_1 will not describe that behavior properly unless under a quite refined element mesh. Besides, the assumption of constant ε implies discontinuous functions for both u_1 and u_2 and its derivatives.

These are some new results using finite-element model techniques, which will be further detailed in a close future.

REFERENCES

André, J.C. and Mazzilli, C.E.N., 1996, "Reduction technique applied to the nonlinear analysis of beams", Second European Nonlinear Oscillations Conference, pp. 57-60, Prague

André, J.C., 1996, "Reduction technique for nonlinear dynamic analysis of beams and planar frames", Proceedings of the Symposium dedicated to the 70th Birthday of Wolf Altman – Recent developments in solid mechanics – pp. 163-170, Rio de Janeiro, Brasil

André, J.C., Mazzilli, C.E.N., and Pereira, D.S., 1999, "Nonlinear Dynamics of Beams: Reduction Mathematical Models", Proceedings of the Pan-American Congress of Applied Mechanics, Jan 9-13, Rio de Janeiro

Soares, M.E.S., 1998, "Modos não-lineares em sistemas discretizados pelo método dos elementos finitos", Doctorate thesis, University of São Paulo, São Paulo, São Paulo, Brazil